The University of Chicago

A RECORDER FOR ELECTRICAL POTENTIALS

THE DAMPING OF PIEZOELECTRIC SYSTEMS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF PHYSICS

1938

By
FRANKLIN OFFNER

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A Recorder for Electrical Potentials

The Damping of Piezoelectric Systems

FRANKLIN OFFNER
Departments of Physics and Physiology, University of Chicago, Chicago, Illinois
(Received August 31, 1939)

I. INTRODUCTION

THE expense, inconvenience, and delay of continuous photography of bioelectric potentials can be eliminated by an ink-writing oscillograph, provided the frequencies to be recorded are within its range. Dr. R. W. Gerard and the writer described a piezoelectric crystal actuated ink writer[1] with range great enough to record many of the potentials encountered in biology. A study of the theory of operation of this instrument has led to several improvements in design.

A diagram of the improved instrument appears in Fig. 1. Two rectangular Rochelle salt plates A, cut as described by Sawyer,[2] are mounted symmetrically. Three corners of each plate are clamped between rubber blocks B and steel ball pivots C. The link D transmits the motion of the fourth corner through the segment E and pulley F to the pen shaft. For strength, rigidity, and lightness, the pen lever G is a Duralumin stamping 6.5 cm long. It supports the ink tube H and writing point J which are fed ink through the small rubber tube K. Friction is minimized by keeping a film of ink between pen and paper. The paper is driven at speeds up to 25 cm/sec.

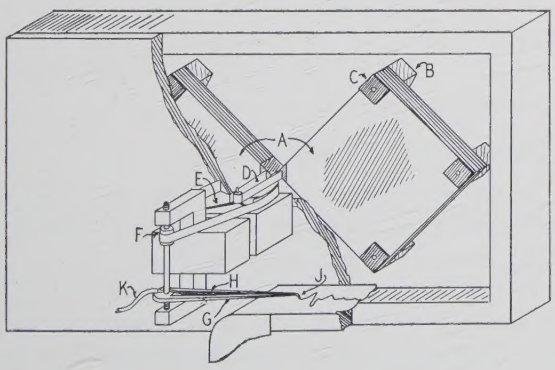


FIG. 1. Diagram of the crystal ink writer.

[1] F. Offner and R. W. Gerard, Science **84**, 209 (1936).
[2] C. B. Sawyer, U. S. Patent 1,803,275. Fig. 13.

The symmetrical placement of the crystals, which are electrically paralleled so that they act mechanically in push-pull, has several advantages. First, the doubling of power increases the product of resonant frequency and maximum displacement by $\sqrt{2}$ (Eq. (1), below). Second, even-order harmonics are canceled; and third, a pure torque is imparted to segment E, thus preventing lost motion due to any play in its bearings. The total motion magnification is 65-fold from driving point to record.

II. THEORY OF THE INK WRITER

a. Units and notation

Mason's derivation[3] of the equivalent electrical circuit of a piezoelectric crystal used as a transducer has been followed in deriving the

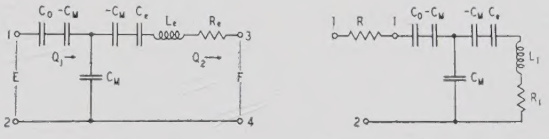


FIG. 2. (a) The electric circuit analog of a piezoelectric transducer. (b) The equivalent circuit of the ink writer.

equivalent circuit of the ink writer. This has facilitated the analysis of the instrument's action.

The electrical analog of a crystal is shown in Fig. 2(a), where 1, 2 represent the electrical terminals; E the applied potential; Q_1 the electrical charge; and 3, 4 the mechanical driving points of the crystals, to which the mechanical load may be connected. Table I shows force, displacement, and velocity of the driving point to be analogous to an electrical potential F, a charge Q_2, and a current i_2, respectively. The compliance C_e is the displacement per dyne applied at 3, 4 with 1, 2 open-circuited. Con-

[3] W. P. Mason, Proc. I. R. E. **23**, 1252 (1935).

versely, the electrostatic capacity C_0 is the charge entering 1, 2 per unit potential developed with 3, 4 open-circuited (i.e., with the crystal clamped). The mutual capacitance-compliance C_m is the ratio of charge to force with 3, 4 open; or the ratio of displacement at 3, 4 to potential developed at 1, 2.

If mechanical dissipation is treated as analogous to an electrical resistance, the latter must in general be a function of i_2. This difficulty is not serious since friction is kept low and will usually be neglected. Fig. 2(b) shows the resultant equivalent circuit of the instrument with the mechanical system connected to the crystal driving points (terminals 3, 4). L_1 is the total mass, as reflected to the driving points; R_1 is the mechanical resistance. The electrical resistance R will be discussed below.

The radian frequency of mechanical resonance is

$$\omega_0 = 1/(L_1 C_e)^{\frac{1}{2}}. \tag{1}$$

If the motion magnification of the mechanical system is increased in the ratio $\phi : 1$, as for example by increasing the pulley to segment diameter ratio, the effective mass of the pen reflected back to the driving points of the crystals is increased in the ratio $\phi^2 : 1$. Because of the relatively minute motion of the remainder of the mechanical system as compared with that of the pen, the mass of the former may be neglected in computing L_1. Thus L_1 is proportional to ϕ^2, and by Eq. (1) $\omega_0\phi$ is constant. Increase in pen excursion obtained by raising the motion magnification therefore results in a proportionate lowering of the resonant frequency.

b. The fidelity problem

Any e.m.f. may be expressed by the Fourier integral

$$E(t) = \int_{-\infty}^{\infty} F(f) e^{j2\pi f t} df$$

and a recorder reproducing all frequencies undistorted would give a facsimile of any wave form. The amplitude-frequency and phase-frequency response curves are thus a criterion of fidelity. These will be discussed in Section II-c.

Another criterion which has been used is the response form for a known signal; e.g., critical damping, which will be discussed in Section II-d.

TABLE I. *Electromechanical units.*

Mechanical Quantity	Electrical Analog	Symbol	Defining Equation
Force	Potential	F	
Displacement	Charge	Q_2	
Velocity	Current	i_2	$i_2 = dQ_2/dt$
Mass	Inductance	L_e	Kinetic Energy $= \frac{1}{2} L_e i_2^2$
Compliance	Capacitance	C_e	$C_e = Q_2/F$
Resistance	Resistance	R_e	
Mutual Capacitance-Compliance		C_m	$C_m = Q_1/F = Q_2/E$
Coefficient of Coupling		k	$k^2 = C_0 C_e / C_m^2$

These two criteria are not equivalent. Consider (for simplicity) a transient $\delta(t) = 0$ for $t \neq 0$, and of unit integrated area.[4] Then

$$E(t) = \delta(t) = \int_{-\infty}^{\infty} e^{j2\pi f t} df$$

and faithful recording of all frequencies would reproduce the original signal. A system cutting off sharply at f_0 gives the recorded wave.

$$\int_{-f_0}^{f_0} e^{j2\pi f t} df = \sin(2\pi f_0 t)/\pi t.$$

i.e., a decrementing sine wave of frequency f_0. On the other hand, a critically damped recorder would give a broadened and shortened, but nonoscillating, spike. Apparently critical damping gives the better reproduction; but if the essential components of the signal lie below f_0, and the damped system fails to transmit some of these, the sharp cut-off system would give the greater accuracy. In practice, the ink writer's cut-off frequency is above the range necessary for many purposes, whereas some desired frequencies would be attenuated by critical damping (e.g. mechanically).

In view of the above, it is well to test a recorder's fidelity by comparison of a signal of the type to be studied with a facsimile (e.g., from a cathode-ray oscillograph).

c. Response to sinusoidal e.m.f.

Some conclusions may be drawn from a qualitative consideration of the equivalent circuit's sinusoidal behavior (Fig. 2(b)).

First, assume potential to be applied at 1, 2 (R out of the circuit). R_1 being assumed small, the effect of it and L_1 may be neglected at low frequencies, and the network appears as a pure

[4] Such a signal is obviously physically unrealizable but would be approached by a very brief spike.

capacitance. The input current i_1 and the resultant velocity i_2 are then both proportional to frequency. But

$$i_2 = dQ_2/dt = \omega Q_{\max} \sin \omega t.$$

so that the resulting excursion is independent of frequency. However, with increasing frequency L_1 appreciably reduces the net capacitive reactance, and i_1 and i_2 both increase more rapidly than the frequency. Thus the pen excursion increases with frequency up to resonance;[5] it diminishes beyond.

An interesting possibility results from the simultaneous increase of i_1 and i_2. Apply the potential at 1′, 2 (R in the circuit). If the value of R is properly chosen, the input impedance of the crystal at low frequencies will be much greater than, and in quadrature to, R; the latter then has little effect on the 1, 2 voltage (i.e., that applied to the crystal). As the frequency increases, the input impedance decreases and becomes more nearly resistive; the fractional voltage drop in R thus increases. At resonance the impedance is a minimum and purely resistive; R is here most effective in reducing the crystal voltage. Thus the drop in the 1, 2 potential parallels the natural increase in deflection with frequency, and a much more nearly constant amplitude results. This effect is verified by a quantitative treatment.

The response of the instrument may be calculated from the mesh equations. If E, the impressed e.m.f., is of radian frequency ω, and R_1 is assumed negligibly small, these equations are

$$pE = i_1(pR + 1/C_0) - i_2(1/C_m). \tag{2}$$

$$0 = -i_1(1/C_m) + i_2(p^2 L_1 + 1/C_e), \tag{3}$$

where $p = j\omega$. Then

$$Q_2 = i_2/p = k^2\omega_0^2 C_m E/[(1-k^2)\omega_0^2 - \omega^2 + j\omega RC_0(\omega_0^2 - \omega^2)]$$

with k, the coefficient of coupling, defined by

$$k^2 = C_0 C_e / C_m^2$$

and, by Eq. (1), $\omega_0^2 = 1/L_1 C_e$. When $\omega = \omega_0$ the second term of the denominator is zero for any R; i.e., $i_1 = 0$, and ω_0 is the anti-resonant frequency.

[5] As shown below, this electrical resonant frequency is $(1-k^2)^{\frac{1}{2}}$ times ω_0, the frequency of mechanical resonance.

If $R = 0$, the amplitude becomes infinite at

$$\omega^2 = (1-k^2)\omega_0^2 = \omega_1^2.$$

ω_1 is the resonant frequency of the circuit. The amplitude factor is then

$$k^2\omega_0^2 C_m / [(\omega_1^2 - \omega^2)^2 + R^2\omega^2 C_0^2(\omega_0^2 - \omega^2)^2]^{\frac{1}{2}} \tag{4}$$

and the phase shift is

$$\tan \delta = R\omega C_0 \frac{\omega_0^2 - \omega^2}{\omega_1^2 - \omega^2}. \tag{5}$$

Thus, plotted against ω/ω_1 the undamped ($R = 0$) phase-shift and (except for a constant factor) amplitude characteristics are independent of k. The undamped and damped response curves as given by Eqs. (4) and (5) are illustrated in Fig. 5(a) and (b); these are described in Section III-a.

Another interpretation of ω_0 and ω_1 is this: if a mechanical transient is applied, ω_0 will be the frequency of oscillation with the electrical terminals open-circuited, and ω_1 that when short-circuited.

d. Transient response

It has been shown that the sinusoidal response is improved by a series electrical resistance. But can the system be damped critically in this way? That is, can the oscillatory terms in the transient response be eliminated by resistance in the first (electrical) mesh alone?

The natural (force-free) modes, p, of the network of Fig. 2(b) (R_1 again negligibly small) are given by the roots of coefficients of the i's in Eqs. (2) and (3); that is, by the roots of

$$p^3 RL_1 + p^2 L_1/C_0 + pR/C_e + 1/C_0 C_e - 1/C_m^2 = 0$$

or

$$p^3 + p^2/RC_0 + p\omega_0^2 + \omega_0^2\lambda/RC_0 = 0 \tag{6}$$

with $\lambda = 1 - k^2$. If the damping is to be critical or greater, all roots of this cubic must be real. The condition for this is that

$$b^2/4 + a^3/27 \leqq 0,$$

where

$$a = (\omega_0^2/3)(3 - 1/x^2),$$
$$b = (\omega_0^3/27x)(2/x^2 + 27\lambda - 9)$$

and

$$x = RC_0\omega_0.$$

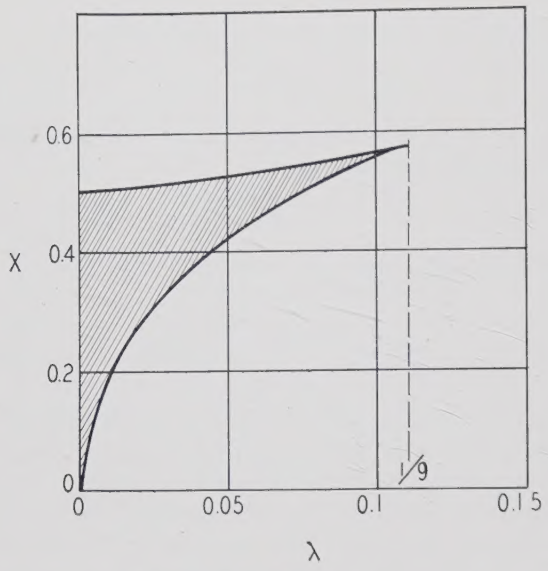


FIG. 3. The damping locus.

Substituting, we have after simplification

$$y=4x^4+3x^2(9\lambda^2-6\lambda-\tfrac{1}{3})+4\lambda\leqq 0. \qquad (7)$$

The values of λ (in the range $1\geqq\lambda\geqq 0$) for which there exist real positive values of x satisfying this inequality are shown by the shaded area of Fig. 3. The boundary curve gives the value of x for critical damping at the corresponding λ, while points enclosed by the curve correspond to over-damping. It is seen that critical damping is possible only when

$$\lambda=1-k^2\leqq\tfrac{1}{9},$$
$$k\geqq 0.943.$$

Thus critically damping by insertion of resistance in the first mesh alone is impossible unless k is greater than 0.943. The coefficient of coupling k is a property of the crystal plate, depending only on the composition, orientation with respect to the crystallographic axes, and temperature. Values for a typical Rochelle salt element (as a function of temperature) are given in Fig. 4, for which the author is indebted to Dr. Mason.[6] Since the highest k attained is approximately 0.83, complete damping by this means should be impossible. Partial experimental verification of this is given in Section III.

[6] W. P. Mason, personal communication.

III. Experimental Tests of Fidelity

Partial confirmation of the foregoing theoretical results has been obtained by several experiments.

a. The equivalent network

To check the circuit theory developed, the electrical network (Fig. 2(b)) was examined oscillographically. Furthermore, it is more practical to investigate the effect of circuit parameters empirically than by a detailed study of the circuit equations.

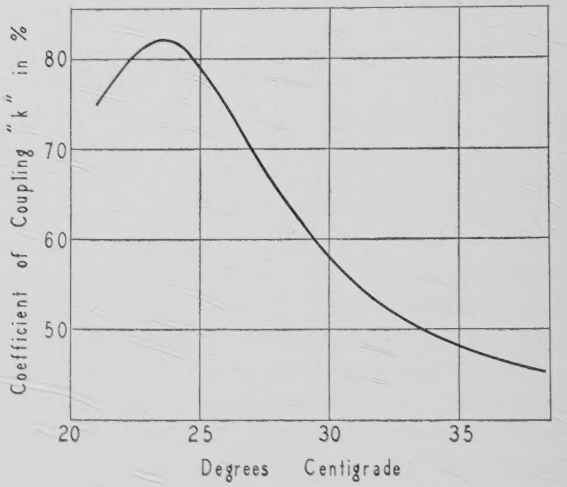


FIG. 4. The variation of the coefficient of coupling of a typical Rochelle salt plate with temperature.

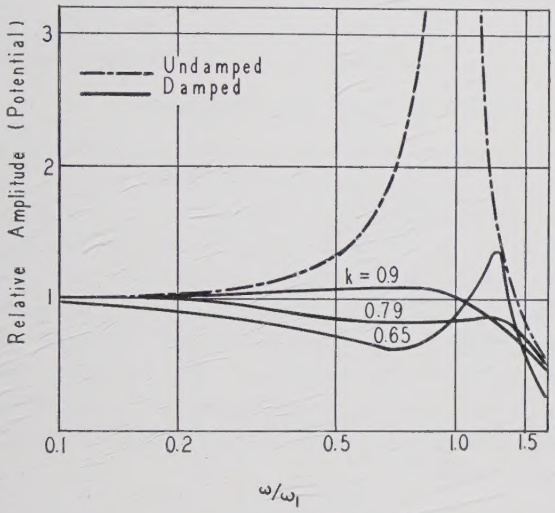
In realizing the circuit, the negative and adjoining positive capacitances were combined into two positive condensers; a cathode-ray oscillograph showed the potential across the condenser formed from C_e and $-C_m$. Since this potential is proportional to Q_2, it is a measure of the displacement of the electromechanical system.

Table II shows the experimental results for the values of k appearing in the first column. The second and third columns list the approximate values of $x=R\omega_0 C_0$ for best transient and sinusoidal fidelity, respectively.

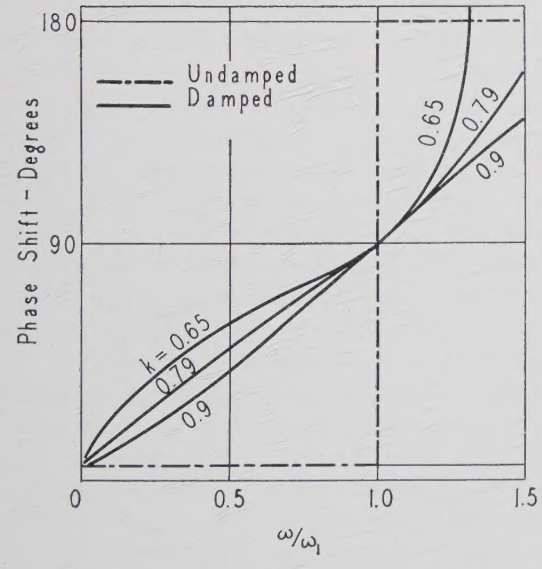
The best frequency response and damping are attained with approximately the same resistance. Fig. 5(a) illustrates the amplitude-frequency, and Fig. 5(b) the phase-frequency characteristics, both with and without damping; the amplitude response has been calculated by Eq. (4), and verified experimentally. The improvement in

linearity effected by the resistor is obvious. The phase shift has been calculated by Eq. (5). With $R=0$, it is zero up to ω_1, where it shifts through 180°. In practice, there is always some dissipation in each mesh, and appreciable phase shift below ω_1; damping then improves the phase response below ω_1 as well.

The apparent critical damping with k as low as 0.9 probably results from the inability to detect a small oscillation in the presence of a large exponential term.

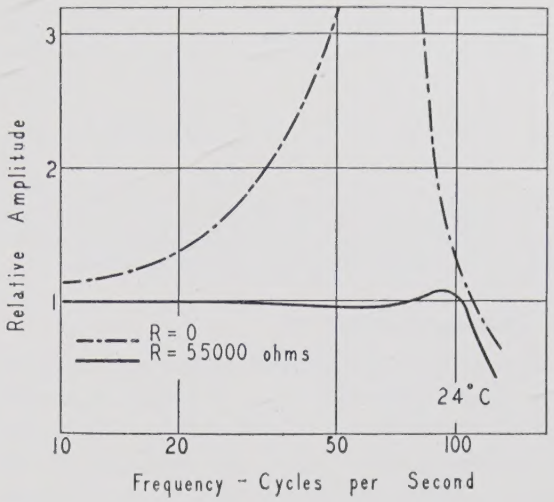


(a)

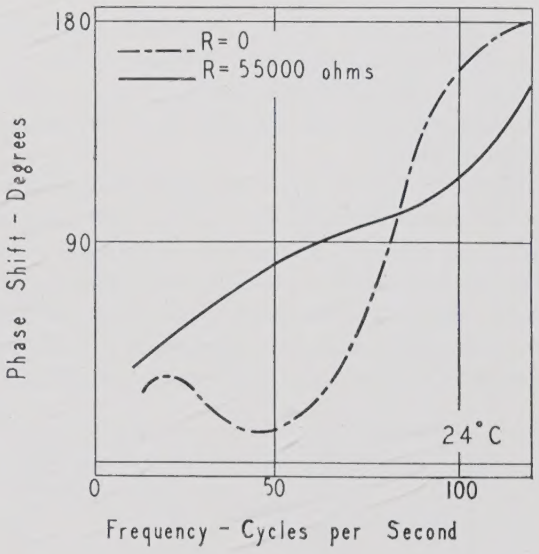


(b)

FIG. 5. Theoretical response characteristics of the equivalent network (Fig. 2(b)). (a) Amplitude-frequency. (b) Phase-frequency.



(a)



(b)

FIG. 6. Experimental response characteristics of the ink writer. (a) Amplitude-frequency. (b) Phase-frequency.

TABLE II. *Response of the equivalent network.*

k	$x=R\omega_0C_0$ FOR TRANSIENT*	$x=R\omega_0C_0$ FOR SINUSOIDAL†	REMARKS
0.90	0.72	0.51	Appears critically damped
0.79	1.2	1.2	Small transient oscillation
0.65	1.9	2.1	Larger transient

* Experimentally determined to give least oscillation.
† Gives most nearly uniform frequency response (Fig. 5(a)).

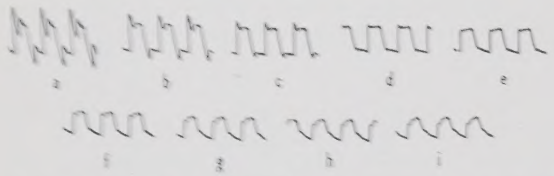


FIG. 7. Electrical damping of transient response. a. $R=0$, b. $R=10{,}000$ ohms, c. $R=20{,}000$ ohms, d. $R=30{,}000$ ohms, e. $R=40{,}000$ ohms, f. $R=50{,}000$ ohms, g. $R=60{,}000$ ohms, h. $R=70{,}000$ ohms, and i. $R=80{,}000$ ohms.

b. Crystal currents

Transient potentials were applied to a Rochelle salt plate, and the input currents examined oscillographically as the damping resistor and temperature were varied. As predicted by the theory, at no time was it possible to eliminate completely the oscillatory components.

c. Response of the instrument

Figure 6 (a) shows the amplitude-frequency and Fig. 6(b) the approximate phase-frequency response of the ink writer described in Section I. The improvement effected by the series resistor is again evident.

Figure 7 illustrates the damping of transients by the input resistor. It may be seen that no value of resistor completely suppresses overshooting (a residual oscillation). Here again, about the same damping gives most faithful transient and frequency response. In practice the resistor is adjusted so that a d.c. pulse and a 60-c.p.s. wave of the same voltage give the same pen excursion.

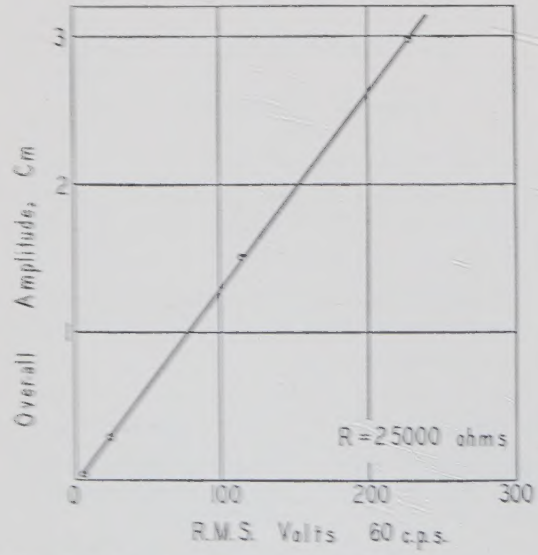


FIG. 8. Voltage-amplitude linearity of the ink writer.

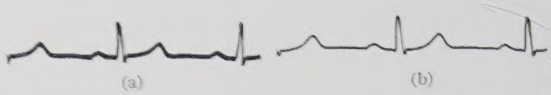


FIG. 9. A human electrocardiogram. (a) Cathode-ray oscillograph record. (b) Ink writer record.

The voltage-amplitude linearity characteristics measured at 60 c.p.s., is shown in Fig. 8. At very low frequencies and large amplitudes Rochelle salt shows a hysteresis effect similar to the magnetism of ferromagnetic materials. This seems to cause no appreciable error in the frequencies usually encountered in biological work.

d. Recording of electrocardiograms

As mentioned before, probably the best test of fidelity of any recorder is comparison of the records obtained with a known wave form of the type to be studied. Electrocardiograms must frequently be recorded in biological work, and since they have a wide range of component frequencies, they have been used to check the accuracy of the instrument. Fig. 9 shows the same human electrocardiogram, as recorded by the cathode-ray oscillograph at *a* and by the ink writer at *b*.

IV. Summary and Acknowledgment

A piezoelectric ink-writing oscillograph has been found useful in the recording of bioelectric potentials. The analysis of the electromechanical system of the instrument is simplified by deriving the equivalent electrical network. From this the phase-frequency, amplitude-frequency, and transient characteristics can be calculated. While these are all improved by inserting a series resistor, critical damping cannot be achieved by this means alone in a piezoelectric device because the necessary coefficient of coupling, greater than 0.943, is unobtainable in a crystal.

Critical damping and best frequency response are not necessarily equivalent. The best test of fidelity is comparison of records with the wave form of the applied potential. Such comparisons have been made with the instrument described.

I wish to express my gratitude to Dr. Carl Eckart for his assistance in the theoretical portion of this investigation, and to Dr. R. W. Gerard for his encouragement.